

Exercise 5

Use power series to solve the differential equation.

$$y'' + xy' + y = 0$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x .

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Differentiate the series with respect to x once more.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x inside the summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Because of n in the summand, the second series can start from $n = 0$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Make the substitution $n = k + 2$ in the first series and the substitution $n = k$ in the second and third series.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^{(k+2)-2} + \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

Simplify the first sum.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

Now that all the sums start from $k = 0$ and have x^k in the summand, they can be combined.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} x^k + k a_k x^k + a_k x^k \right] = 0$$

Simplify the summand.

$$\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} + (k+1)a_k] x^k = 0$$

Since x^k isn't zero, the quantity in square brackets must be zero.

$$(k+2)(k+1)a_{k+2} + (k+1)a_k = 0$$

Solve for a_{k+2} .

$$a_{k+2} = -\frac{1}{k+2}a_k$$

In order to determine a_k , plug in values for k and try to find a pattern.

$$k = 0: a_2 = -\frac{1}{0+2}a_0 = -\frac{a_0}{2}$$

$$k = 1: a_3 = -\frac{1}{1+2}a_1 = -\frac{a_1}{3}$$

$$k = 2: a_4 = -\frac{1}{2+2}a_2 = -\frac{1}{4} \left(-\frac{a_0}{2} \right) = (-1)^2 \frac{a_0}{2 \cdot 4}$$

$$k = 3: a_5 = -\frac{1}{3+2}a_3 = -\frac{1}{5} \left(-\frac{a_1}{3} \right) = (-1)^2 \frac{a_1}{3 \cdot 5}$$

$$k = 4: a_6 = -\frac{1}{4+2}a_4 = -\frac{1}{6} \left[(-1)^2 \frac{a_0}{2 \cdot 4} \right] = (-1)^3 \frac{a_0}{2 \cdot 4 \cdot 6}$$

$$k = 5: a_7 = -\frac{1}{5+2}a_5 = -\frac{1}{7} \left[(-1)^2 \frac{a_1}{3 \cdot 5} \right] = (-1)^3 \frac{a_1}{3 \cdot 5 \cdot 7}$$

⋮

The general formula for the even subscripts is

$$a_{2m} = (-1)^m \frac{a_0}{(2m)!!} = (-1)^m \frac{a_0}{2^m m!},$$

and the general formula for the odd subscripts is

$$a_{2m+1} = (-1)^m \frac{1}{(2m+1)!!} a_1 = (-1)^m \frac{2^m m!}{(2m+1)!} a_1.$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} a_{2m} x^{2m} + \sum_{m=0}^{\infty} a_{2m+1} x^{2m+1} \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{a_0}{2^m m!} x^{2m} + \sum_{m=0}^{\infty} (-1)^m \frac{2^m m!}{(2m+1)!} a_1 x^{2m+1}, \end{aligned}$$

where a_0 and a_1 are arbitrary constants.